

QUATERNIONS AND KUDLA'S MATCHING PRINCIPLE

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ABSTRACT. In this paper, we prove some interesting identities, among average representation numbers (associated to definite quaternion algebras) and ‘degree’ of Hecke correspondences on Shimura curves (associated to indefinite quaternion algebras).

1. INTRODUCTION

In this paper, we prove some interesting identities relating two different quaternion algebras using Kudla’s matching principle [Ku2, Section 4].

Let D be a square square free integer, and let $B = B(D)$ be the unique quaternion algebra of discriminant D over \mathbb{Q} , i.e., B is ramified at a finite prime p if and only if $p|D$. The reduced norm, denoted by \det in this paper, gives a canonical quadratic form Q on B and makes it a quadratic space $V = (B, \det)$. For a positive integer N prime to D , let $\mathcal{O}_D(N)$ be an Eichler order of B of conductor N , which is an even integral lattice of V , and denote by L . The quaternion B is definite if and only if D has odd number of prime factors. When B is definite, it is a very interesting and hard question to compute the representation number (for a positive integer m)

$$r_L(m) = |\{x \in \mathcal{O}_D(N) : \det x = m\}|.$$

On the other hand, the average over the genus $\text{gen}(L)$, which we denote by

$$r_{D,N}(m) = r_{\text{gen}(L)}(m) = \left(\sum_{L_1 \in \text{gen}(L)} \frac{1}{|\text{Aut}(L_1)|} \right)^{-1} \sum_{L_1 \in \text{gen}(L)} \frac{r_{L_1}(m)}{|\text{Aut}(L_1)|}$$

is product of so-called local densities, thanks to Siegel’s seminal work in 1930’s [Si], which are computable (see example [Ya1]). We remark that $\text{gen}(L)$ consists of (equivalence classes) of right ideals of all maximal orders when $N = 1$. Notice that $r_{D,N}$ depends only on D and N and is independent of the choice of Eichler orders of conductor N . Using Kudla’s matching principle ([Ku2], see also Section 2), we will prove the following theorem in Section 4.

Theorem 1.1. *Let D be a square-free positive integer with even number of prime factors, let $p \neq q$ be two different primes not dividing D , and let N be a positive integer prime to*

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Dpq . Then

$$-\frac{2}{q-1}r_{Dp,N}(m) + \frac{q+1}{q-1}r_{Dp,Nq}(m) = -\frac{2}{p-1}r_{Dq,N}(m) + \frac{p+1}{p-1}r_{Dq,Np}(m)$$

for every positive integer m .

We remark that $r_{p,N}(m)$ has geometric interpretations. For example, Gross and Keating ([GK], [Wed, Page 27]) proves

$$r_{p,1}(m) = 2 \left(\sum_E \frac{1}{|\text{Aut}(E)|} \right)^{-2} \sum_{(E,E')} \frac{r_{\text{Hom}(E,E')}(m)}{|\text{Aut}(E)||\text{Aut}(E')|}.$$

Here the sums are over supersingular elliptic curves over $\bar{\mathbb{F}}_p$, and $\text{Hom}(E, E')$ is the quadratic lattice of isogenies (from E to E') with degree as the quadratic form. Replacing E by a pair (E, C) where C is a cyclic subgroup of E of order N , and $\text{Hom}(E, E')$ by $\text{Hom}((E, C), (E', C'))$, one gets $r_{p,N}(m)$ (see Section 4 for detail). So Theorem 1.1 gives some relations between supersingular elliptic curves over different primes when we take $D = 1$. We also remark that $|\text{Aut}(E)|$ has a simple formula (see [Gr])

When $B(D)$ is indefinite, the representation number does not make sense anymore as a number can be represented infinitely many times. In this case, the geometry of Shimura curves comes in. We fix an embedding of $i : B(D) \hookrightarrow M_2(\mathbb{R})$ such that $B(D)^\times$ is invariant under the automorphism $x \mapsto x^* = {}^t x^{-1}$ of $\text{GL}_2(\mathbb{R})$. Let $\Gamma_0^D(N) = \mathcal{O}_D(N)^1$ be the group of (reduced) norm 1 elements in $\mathcal{O}_D(N)$ and let $X_0^D(N) = \Gamma_0^D(N) \backslash \mathbb{H}$ be the associated Shimura curve. For a positive integer m , let $T_{D,N}(m)$ be the Hecke correspondence on $X_0^D(N)$ defined by

$$(1.1) \quad T_{D,N}(m) = \{([z_1], [z_2]) \in X_0^D(N) \times X_0^D(N) : z_1 = i(x)z_2 \text{ for some } x \in \mathcal{O}_D(N), \det x = m\}.$$

Define $\deg T_{D,N}(m) = \deg(T_{D,N}(m) \rightarrow X_0^D(N))$ under the projection $([z_1], [z_2]) \mapsto [z_1]$.

Let $\Omega_0 = -\frac{1}{4\pi}y^{-2}dx \wedge dy$ be the volume form on $X_0^D(N)$ (associated to $\Omega_{X_0^D(N)}^\vee$), and let

$$\text{vol}(X_0^D(N), \Omega_0) = \int_{X_0^D(N)} \Omega_0$$

be the volume of $X_0^D(N)$ with respect to Ω_0 , which is a positive rational number (see (5.3)).

Finally, we define (when D has even number of prime factors)

$$(1.2) \quad r'_{D,N}(m) = \frac{1}{\text{vol}(X_0^D(N), \Omega_0)} \deg T_{D,N}(m).$$

Similar to Theorem 1.1, we will prove the following theorems in Section 5.

Theorem 1.2. *Let D be a square-free positive integer with odd number of prime factors, let $p \neq q$ be two different primes not dividing D , and let N be a positive integer prime to Dpq . Then*

$$-\frac{2}{q-1}r'_{Dp,N}(m) + \frac{q+1}{q-1}r'_{Dp,Nq}(m) = -\frac{2}{p-1}r'_{Dq,N}(m) + \frac{p+1}{p-1}r'_{Dq,Np}(m)$$

for every positive integer m .

Theorem 1.3. *Let D be a square-free positive integer with odd number of prime factors, let $p \nmid D$ be a prime, and let N be a positive integer prime to Dp . Then*

$$r'_{Dp,N}(m) = -\frac{2}{p-1}r_{D,N}(m) + \frac{p+1}{p-1}r_{D,Np}(m).$$

Theorem 1.4. *Let $D > 1$ be a square-free positive integer with even number of prime factors, let $p \nmid D$ be a prime, and let N be a positive integer prime to Dp . Then*

$$r_{Dp,N}(m) = -\frac{2}{p-1}r'_{D,N}(m) + \frac{p+1}{p-1}r'_{D,Np}(m).$$

This paper is organized as follows. In Section 2, we review Weil representation and Kudla's matching principle in general case. In Section 3, we prove some local matching between division and matrix quaternion algebras over a local field. In Section 4 we look at two global quaternions different at two primes carefully and prove Theorem 1.1. In Section 5, we associate product of two Shimura curves to the quadratic space coming from an indefinite quaternion and compute the theta integral via 'degree' of Hecke correspondence, and prove Theorems 1.2, 1.3 and 1.4.

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2. PRELIMINARIES AND KUDLA'S MATCHING PRINCIPLE

Let $\psi : \mathbb{A}/\mathbb{Q} \rightarrow \mathbb{C}$ be the canonical unramified additive character, such that $\psi_\infty(x) = e^{2\pi i x}$. Let (V, Q) be a nondegenerate quadratic space over \mathbb{Q} of even dimension m with the quadratic form Q , and let

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det V)_\mathbb{A}$$

be the associated quadratic character. Let $\omega = \omega_{\psi,V}$ be the associated Weil representation of $O(V)(\mathbb{A}) \times SL_2(\mathbb{A})$ on $S(V(\mathbb{A}))$, where $S(V(\mathbb{A}))$ is the Schwartz-Bruhat function space. The orthogonal group $O(V)(\mathbb{A})$ acts on $S(V(\mathbb{A}))$ linearly,

$$\omega(h)\varphi(x) = \varphi(h^{-1}x).$$

The $SL_2(\mathbb{A})$ -action is determined by (see for example [Ku1])

$$\begin{aligned} \omega(n(b))\varphi(x) &= \psi(bQ(x))\varphi(x), \\ (2.1) \quad \omega(m(a))\varphi(x) &= \chi_V(x) |a|^{\frac{m}{2}} \varphi(ax), \\ \omega(w)\varphi &= \gamma(V)\widehat{\varphi} = \gamma(V) \int_{V(\mathbb{A})} \varphi(y)\psi((x,y))dy, \end{aligned}$$

where for $a \in \mathbb{A}^\times$, $b \in \mathbb{A}$

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

dy is the Haar measure on $V(\mathbb{A})$ self-dual with respect to $\psi((x, y))$, and $\gamma(V)$ is a 8-th root of unity (Weil index). Similarly, one has Weil representation at each prime p , which we still denote ω if there is no confusion. Let $P = NM$ be the standard Borel subgroup of SL_2 , where N and M are subgroups of $n(b)$ and $m(a)$ respectively. It is well-known that the theta kernel ([We])

$$(2.2) \quad \theta(g, h, \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g) \varphi(h^{-1}x), \quad \varphi \in S(V(\mathbb{A}))$$

is left $O(V)(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q})$ -invariant and is thus an automorphic form on $[O(V)] \times [\mathrm{SL}_2]$. Here denote $[G] = G(\mathbb{Q}) \backslash G(\mathbb{A})$ for an algebraic group G over \mathbb{Q} . So the theta integral

$$(2.3) \quad I(g, \varphi) = \frac{1}{\mathrm{vol}([O(V)])} \int_{[O(V)]} \theta(g, h, \varphi) dh$$

is an automorphic form on $[\mathrm{SL}_2]$ if the integral is absolutely convergent, which is the case precisely when V is anisotropic or $\dim(V) - r > 2$, where r is the Witt index of V . There is another way to construct automorphic forms from $\phi \in S(V(\mathbb{A}))$ via Eisenstein series.

For $s \in \mathbb{C}$, let $I(s, \chi_V)$ be the principal series representation of $\mathrm{SL}_2(\mathbb{A})$ consisting of smooth functions $\Phi(s)$ on $\mathrm{SL}_2(\mathbb{A})$ such that

$$(2.4) \quad \Phi(nm(a)g, s) = \chi_V(a)|a|^{s+1}\Phi(g, s).$$

There is a $\mathrm{SL}_2(\mathbb{A})$ -intertwining map ($s_0 = \frac{m}{2} - 1$)

$$(2.5) \quad \lambda = \lambda_V : S(V(\mathbb{A})) \rightarrow I(s_0, \chi_V), \quad \lambda(\varphi)(g) = \omega(g)(0).$$

Let $K_\infty K$ be the subgroup $SO_2(\mathbb{R}) \times \mathrm{SL}_2(\hat{\mathbb{Z}})$ in $\mathrm{SL}_2(\mathbb{A})$. A section $\Phi(s) \in I(s, \chi)$ is called standard if its restriction to $K_\infty K$ is independent of s . By Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K_\infty K$, the function $\lambda(\varphi) \in I(s_0, \chi)$ has a unique extension to a standard section $\Phi(s) \in I(s, \chi)$, where $\Phi(s_0) = \lambda(\varphi)$. The Eisenstein series is given by

$$(2.6) \quad E(g, s, \varphi) = \sum_{\gamma \in P \backslash \mathrm{SL}_2(\mathbb{Q})} \Phi(\gamma g, s).$$

When V is anisotropic or that $\dim(V) - r > 2$, Kudla and Rallis ([KR1] [KR2]) proved that the Eisenstein series is holomorphic at $s = s_0$ (extending Weil's classical work [We]) and produces an automorphic form $[\mathrm{SL}_2]$. The two ways (theta integral and Eisenstein series) give the same automorphic form—the well-known Siegel-Weil formula as extended by Kudla and Rallis ([KR1], [KR2]).

Theorem 2.1. (*Siegel-Weil formula*) Assume that V is anisotropic or that $\dim(V) - r > 2$, where r is the Witt index of V , so that the theta integral is absolutely convergent. Then $E(g, s; \Phi)$ is holomorphic at the point $s_0 = m/2 - 1$, where $m = \dim(V)$, and

$$E(g, s_0, \Phi) = \kappa I(g, \varphi),$$

where $\kappa = 2$ when $m \leq 2$ and $\kappa = 1$ otherwise.

Let $V^{(1)}, V^{(2)}$ be two quadratic spaces with the same dimension and the same character χ , then there is a following graph

$$(2.7) \quad \begin{array}{ccc} S(V^{(1)}(\mathbb{A})) & \xrightarrow{\lambda_{V^{(1)}}} & \\ & \searrow & \\ & I(s_0, \chi) & \\ & \nearrow & \\ S(V^{(2)}(\mathbb{A})) & \xrightarrow{\lambda_{V^{(2)}}} & \end{array} .$$

Following Kudla [Ku2], we make the following definition.

Definition 2.2. For an prime $p \leq \infty$, $\varphi_p^{(i)} \in S(V_p^{(i)})$ are said to be matching if

$$\lambda_{V_p^{(1)}}(\varphi_p^{(1)}) = \lambda_{V_p^{(2)}}(\varphi_p^{(2)}).$$

$\varphi^{(i)} = \prod_p \varphi_p^{(i)} \in S(V^{(i)}(\mathbb{A}))$ are said to be matching if they match at each prime p .

By the Siegel-Weil formula, we have the following Kudla matching principle ([Ku2]): Under the assumption of Theorem 2.1 for both $V^{(1)}$ and $V^{(2)}$, one has, for a matching pair $(\varphi^{(1)}, \varphi^{(2)})$,

$$(2.8) \quad I(g, \varphi^{(1)}) = I(g, \varphi^{(2)}).$$

An lattice L of V is a free \mathbb{Z} -submodule of V of rank m . It is even integral if $Q(x) \in \mathbb{Z}$ for every $x \in L$. We define the dual of L as

$$L^\# = \{x \in V : (x, L) \subset \mathbb{Z}\}.$$

Locally,

$$L_p^\# = \{x \in V_p : (x, L_p) \subset \mathbb{Z}_p\}.$$

L_p is called self-dual if $L_p^\# = L_p$.

3. MATCHINGS ON QUATERNIONS

Over a local field \mathbb{Q}_p , there are two quaternions: the matrix algebra $B^{sp} = M_2(\mathbb{Q})$ (split quaternion) and the unique division quaternion B^{ra} (ramified quaternion). Let $V = V^{sp}$ or V^{ra} be the associated quadratic space with reduced norm as the quadratic form $\det(x) = xx^\iota$, where x^ι is the main involution on quaternion algebra B . Both spaces have trivial quadratic character χ_V . So we have $\mathrm{SL}_2(\mathbb{Q}_p)$ -intertwining operators

$$\lambda : S(V) \rightarrow I(1), \varphi \mapsto \lambda(\varphi) = \omega_V(g)\varphi(0).$$

Here $I(s) = I(s, \text{trivial})$. We will use superscript ra and sp to indicate the association with division or matrix quaternion algebra. It is known ([Ku2]) that λ^{sp} is surjective while the image of λ^{ra} is of codimension 1. So every function φ^{ra} has some matching in $S(V^{sp})$. The purpose of this section is to give some explicit matchings and obtain some interesting global identities. In next section, we will give arithmetic and geometric interpretations of these identities in special cases.

3.1. The finite prime case $p < \infty$. We assume $p < \infty$ in this subsection. Let $L^{ra} = \mathcal{O}_{B^{ra}}$ be the maximal order of $B^{ra} = B_p^{ra}$, which consists of all elements of B whose reduced norm is in \mathbb{Z}_p . We don't use the subscript p for simplicity in this subsection. Let $L_0^{sp} = M_2(\mathbb{Z}_p)$ and

$$L_1^{sp} = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : c \equiv 0 \pmod{p}\}.$$

Then

$$L^{ra,\sharp} = \pi^{-1} L^{ra}, \quad L_1^{sp,\sharp} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}_p) : a, c, d \in \mathbb{Z}_p, b \in \frac{1}{p}\mathbb{Z}_p \right\}.$$

Here $\pi \in B^{ra}$ is a 'uniformizer', i.e., $\pi^\iota = -\pi$ and $\pi^2 = p$. We denote

$$\varphi^{ra} = \text{char}(L^{ra}), \quad \varphi^{ra,\sharp} = \text{char}(L^{ra,\sharp}),$$

and

$$(3.1) \quad \varphi_i^{sp} = \text{char}(L_i^{sp}), \quad i = 0, 1, \quad \text{and} \quad \varphi_2^{sp} = \text{char}(L_1^{sp,\sharp}).$$

Proposition 3.1. *Let the notation be as above. Then*

- (1) $\varphi^{ra} \in S(V^{ra})$ matches with $\frac{-2}{p-1}\varphi_0^{sp} + \frac{p+1}{p-1}\varphi_1^{sp} \in S(V^{sp})$.
- (2) $\varphi^{ra,\sharp} \in S(V^{ra})$ matches with $\frac{2p}{p-1}\varphi_0^{sp} - \frac{p+1}{p-1}\varphi_2^{sp} \in S(V^{sp})$.

Proof. (1) Since

$$\text{SL}_2(\mathbb{Z}_p) = K_0(p) \cup N(\mathbb{Z}_p)wK_0(p), \quad K_0(p) = L_1^{sp} \cap \text{SL}_2(\mathbb{Z}_p),$$

$I(1)^{K_0(p)}$ has dimension 2, and $\Phi \in I(1)^{K_0(p)}$ is determined by $\Phi(1)$ and $\Phi(w)$. Notice that $K_0(p)$ is generated by $n(b)$ and $n_-(c) = w^{-1}n(-c)w$, $b \in \mathbb{Z}_p$ and $c \in p\mathbb{Z}_p$. Using this, one can check that φ^{ra} , φ_i^{sp} are all $K_0(p)$ under respective Weil representation. We check $\omega(n_-(c))\varphi^{ra} = \varphi^{ra}$ and leave others to the reader. One has

$$\omega^{ra}(w)\varphi^{ra}(x) = \gamma(V^{ra})\varphi^{ra,\sharp}(x) \text{vol}(L^{ra}).$$

So

$$\omega^{ra}(n(-c)w)\varphi^{ra}(x) = \gamma(V^{ra}) \text{vol}(L^{ra})\psi_p(-c \det(x))\varphi^{ra,\sharp}(x) = \gamma(V^{ra})\varphi^{ra,\sharp}(x) \text{vol}(L^{ra}),$$

i.e.,

$$\omega^{ra}(n(-c)w)\varphi^{ra} = \omega^{ra}(w)\varphi^{ra}.$$

So

$$\omega^{ra}(n_-(c))\varphi^{ra} = \omega^{ra}(w^{-1})\omega^{ra}(n(-c)w)\varphi^{ra} = \varphi^{ra}$$

as claimed.

Now we have $\lambda^{ra}(\varphi^{ra})$, $\lambda^{sp}(\varphi_i^{sp}) \in I(1)^{K_0(p)}$. Direct calculation gives

$$\begin{aligned} \lambda^{ra}(\varphi^{ra})(1) &= 1, & \lambda^{ra}(\varphi^{ra})(w) &= \gamma(V^{ra})p^{-1} \\ \lambda^{sp}(\varphi_0^{sp}) &= 1, & \lambda^{sp}(\varphi_0^{sp})(w) &= \gamma(V^{sp}), \\ \lambda^{sp}(\varphi_1^{sp}) &= 1, & \lambda^{sp}(\varphi_0^{sp})(w) &= \gamma(V^{sp})p^{-1}. \end{aligned}$$

Since $\gamma(V^{sp}) = -\gamma(V^{ra})(=1)$, one has

$$\lambda^{ra}(\varphi^{ra}) = \frac{-2}{p-1}\varphi_0^{sp} + \frac{p+1}{p-1}\varphi_1^{sp}.$$

This proves (1). Claim (2) is similar and is left to the reader. One just needs to replace $K_0(p)$ by

$$K_0^+(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : b \equiv 0 \pmod{p} \right\}.$$

□

To find matching pairs of coset functions $\varphi_\mu^{ra} = \mathrm{char}(\mu + L^{ra})$, we first need to label them. Let k be the unique unramified quadratic field extension of \mathbb{Q}_p in B^{ra} , and let $\mathcal{O}_k = \mathbb{Z}_p + \mathbb{Z}_p u$ be the ring of integers of k with $u \in \mathcal{O}_k^\times$. Then there is a uniformizer π of B such that $\pi r = \bar{r}\pi$ for $r \in k$ and $\pi^2 = p$. One has then

$$L^{ra} = \mathcal{O}_{B^{ra}} = \mathcal{O}_k + \mathcal{O}_k \pi = \mathbb{Z}_p + \mathbb{Z}_p u + \mathbb{Z}_p \pi + \mathbb{Z}_p u \pi.$$

So one has an isomorphism

$$(Z/p)^2 \cong L^{ra,\sharp}/L^{ra}, \quad (i, j) \mapsto \mu_{i,j}^{ra} = \frac{i + j\mu}{\pi}.$$

Using the identification, we denote $\varphi_{i,j}^{ra}$ for $\mathrm{char}(\mu_{i,j}^{ra} + L^{ra})$. Similarly, we use $\varphi_{i,j}^{sp}$ to denote $\mathrm{char}(\mu_{i,j}^{sp} + L_1^{sp})$, where $\mu_{i,j}^{sp} = \begin{pmatrix} 0 & i \\ i & p \end{pmatrix}$. Let

$$K(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : a-1 \equiv d-1 \equiv b \equiv c \equiv 0 \pmod{p} \right\}.$$

Since

$$(3.2) \quad N(\mathbb{Z}_p)M(\mathbb{Z}_p) \backslash \mathrm{SL}_2(\mathbb{Z}_p)/K(p) = \{1, wn(j), 0 \leq j \leq p-1\},$$

one has $\dim I(1)^{K(p)} = p+1$.

Lemma 3.2. *Let the notation be as above. Then*

- (1) *One has $\varphi_{i,j}^{sp}, \varphi_{i,j}^{ra} \in I(1)^{K(p)}$.*
- (1) *When $ab \equiv cd \pmod{p}$ and $(a, b), (c, d) \neq (0, 0)$, one has $\lambda^{sp}(\varphi_{a,b}^{sp}) = \lambda^{sp}(\varphi_{c,d}^{sp})$.*
- (2) *The set $\{\lambda^{sp}(\varphi_0^{sp}), \lambda^{sp}(\varphi_{1,j}^{sp}), 0 \leq j \leq p-1\}$ gives a basis of $I(1)^{K(p)}$.*

Proof. Claim (1) follows from definition and the fact that $K(p)$ is generated by $n(pb)$ and $n_-(pb)$, $b \in \mathbb{Z}_p$. The invariant under $n(pb)$ is clear. The invariant under $n_-(pb)$ can be verified the same way as in the proof of Proposition 3.1.

- (2) By (3.2), it is only need to check the values at $\{1, wn(i), 0 \leq i \leq p-1\}$.

$$\begin{aligned} \lambda^{sp}(\varphi_{a,b}^{sp})(wn(i)) &= \int_{\mu_{a,b} + \mathcal{O}_p} \psi_p(i \det(x)) dx \\ &= \frac{1}{p} e(abi/p) \end{aligned}$$

and

$$\lambda^{sp}(\varphi_{a,b}^{sp})(1) = 0,$$

where $e(x) = e^{2\pi\sqrt{-1}x}$. The result follows.

(3) By (3.2), we see that $\Phi \in I(1)^{K(p)}$ is determined by the values at

$$\{1, wn(i), 0 \leq i \leq p-1\}.$$

Suppose

$$a\lambda^{sp}(\varphi_0^{sp}) + \sum_{0 \leq j \leq p-1} a_j \lambda^{sp}(\varphi_{1,j}^{sp}) = 0,$$

where $a, a_j \in \mathbb{C}$.

Taking the value at 1:

$$a\lambda^{sp}(\varphi_0^{sp})(1) + \sum_{0 \leq j \leq p-1} a_j \lambda^{sp}(\varphi_{1,j}^{sp})(1) = 0,$$

we get $a=0$. Now we consider the equations

$$\sum_{0 \leq j \leq p-1} a_j \lambda^{sp}(\varphi_{1,j}^{sp})(wn(i)) = 0.$$

The coefficient matrix is $\frac{1}{p}\mathbf{A} := [\lambda^{sp}(\varphi_{1,j}^{sp})(wn(i))]_{0 \leq i, j \leq p-1}$. Since

$$\lambda^{sp}(\varphi_{1,j}^{sp})(wn(i)) = \frac{1}{p}e(ij/p),$$

one sees $\mathbf{A} = (e(ij/p))_{0 \leq i, j \leq p-1}$ and $\det(\mathbf{A}) = \det((e(ij/p))) \neq 0$.

Hence $a = 0, a_j = 0, 0 \leq j \leq p-1$, and $\lambda^{sp}(\varphi_0^{sp}), \lambda^{sp}(\varphi_{1,j}^{sp}), 0 \leq j \leq p-1$ are linear independent. \square

Proposition 3.3. Assume $(k, l) \neq 0$. Let $\mathbf{A} = (e(\frac{ij}{p}))_{0 \leq i, j \leq p-1}$ be the matrix in the proof of Lemma 3.2, and let \mathbf{A}_j be the matrix obtained by replacing j -th column of \mathbf{A} by column $\{-e(\frac{-id_{k,l}}{p}), 0 \leq i \leq p-1\}$, where

$$d_{k,l} = k^2 + kl \operatorname{Tr}_{k/\mathbb{Q}_p}(u) + l^2 N_{k/\mathbb{Q}_p}(u).$$

Then $\sum_{j=0}^{p-1} c_{k,l}(j) \varphi_{1,j}^{sp} \in S(V^{sp})$ is matching with $\varphi_{k,l}^{ra}$, where $c_{k,l}(j) = \frac{\det \mathbf{A}_j}{\det \mathbf{A}}$.

Proof. Since $\lambda^{ra}(\varphi_{k,l}^{ra}) \in I(1)^{K(p)}$, it suffices to check the identity at $\{1, wn(i), 0 \leq i \leq p-1\}$.

Suppose $(k, l) \neq 0$, let

$$\lambda^{ra}(\varphi_{k,l}^{ra}) = b_{k,l} \lambda^{sp}(\varphi_0^{sp}) + \sum_{0 \leq j \leq p-1} c_{k,l}(j) \lambda^{sp}(\varphi_{1,j}^{sp}),$$

where $b_{k,l}, c_{k,l}(j) \in \mathbb{C}$. Taking the value at 1, one gets $b_{k,l} = 0$.

Taking the value at $\{wn(i), 0 \leq i \leq p-1\}$, one has

$$\lambda^{ra}(\varphi_{k,l}^{ra})(wn(i)) = \sum_{0 \leq j \leq p-1} c_{k,l}(j) \lambda^{sp}(\varphi_{1,j}^{sp})(wn(i)).$$

It is easy to check that

$$\lambda^{ra}(\varphi_{k,l}^{ra})(wn(i)) = -\frac{1}{p}e(-\frac{i}{p}(k^2 + kl\text{Tr}(u) + l^2N(u))).$$

Denoting

$$d_{k,l} = k^2 + kl\text{Tr}_{k/\mathbb{Q}_p}(u) + l^2N_{k/\mathbb{Q}_p}(u),$$

then $\lambda^{ra}(\varphi_{k,l}^{ra})(wn(i)) = -\frac{1}{p}e(-\frac{id_{k,l}}{p})$. From the proof of Lemma 3.2, it is known that $\lambda^{sp}(\varphi_{1,j}^{sp})(wn(i)) = \frac{1}{p}e(ij/p)$. So we get $c_{k,l}(j) = \frac{\det \mathbf{A}_j}{\det \mathbf{A}}$. \square

3.2. The case $p = \infty$. In this subsection we consider the case $\mathbb{Q}_p = \mathbb{R}$ and recall a matching pair given in [Ku2]. Notice that B^{ra} in this case is the Hamilton division algebra, and V^{ra} has signature $(4, 0)$. Let $\varphi_\infty^{ra}(x) = e^{-2\pi \det(x)} \in S(V^{ra})$, then φ_∞^{ra} is of weight 2 in the sense

$$\omega^{ra}(k_\theta)\varphi_\infty^{ra} = e^{2i\theta}\varphi_\infty^{ra}, \quad k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

On the other hand, Kudla constructed a family of weight 2 Schwartz function $\varphi_\infty^{sp} \in S(V^{sp})$ as follows [Ku2, Section 4.8]. Recall $V^{sp} = M_2(\mathbb{R})$. Given an orthogonal decomposition

$$(3.3) \quad V^{sp} = V^+ \oplus V^-, \quad x = x^+ + x^-,$$

with V^+ of signature $(2, 0)$ and V^- of signature $(0, 2)$. One defines (Kudla used the notation $\tilde{\phi}(x, z)$)

$$\varphi_\infty^{sp}(x, V^-) = (4\pi(x^+, x^+) - 1)e^{-\pi(x^+, x^+) + \pi(x^-, x^-)}.$$

Kudla proved the following proposition [Ku2, Section 4.8].

Proposition 3.4. *For any orthogonal decomposition as in (3.3), $(\varphi_\infty^{ra}, \varphi_\infty^{sp}(x, V^-))$ is a matching pair, and their (same) image in $I(1)$ is the unique weight 2 section Φ_∞^2 given by*

$$\Phi_\infty^2(n(b)m(a)k_\theta) = |a|^2 e^{2i\theta}.$$

Because of this proposition, we will simply use φ_∞^{sp} for $\varphi_\infty^{sp}(, V^-)$.

3.3. Global matching. For a square-free positive integer, and let $B(D)$ be the unique quaternion algebra over \mathbb{Q} of discriminant D as in the introduction. It is indefinite (i.e., $B(D)_\mathbb{R} \cong M_2(\mathbb{R})$) if and only if D has even number of prime factors and it is $M_2(\mathbb{Q})$, i.e., $D = 1$, precisely when it represents 0. In this paper, we assume $D > 1$ so the Siegel-Weil formula applies. The following matching theorem is clear from Kudla's matching principle (2.8) and Propositions 3.1, 3.3, and 3.4.

Proposition 3.5. *Let $V(D_i)$ be the quadratic spaces associated to the quaternion algebras $B(D_i)$ (with reduced norm as the quadratic form) with square-free integers $D_i > 1$, $i = 1, 2$.*

Assume $\varphi^{(i)} = \prod_p \varphi_p^{(i)} \in S(V(D_i)(\mathbb{A}))$ satisfy the following conditions:

- (1) *When $p = \infty$, $\varphi_\infty^{(i)}$ is φ_∞^{sp} or φ_∞^{ra} depending on whether $B(D_i)_\infty$ is split or non-split.*
- (2) *When $p \nmid D_1 D_2 \infty$ or $p \mid \gcd(D_1, D_2)$, we identify $V(D_1)_p = V(D_2)_p$ and take any $\varphi_p^{(1)} = \varphi_p^{(2)} \in S(V(D_1)_p)$.*
- (3) *When $p \mid \text{lcm}(D_1, D_2)$ but $p \nmid \gcd(D_1, D_2)$, one of $B(D_i)$ is B_p^{sp} and the other is B_p^{ra} , we take $(\varphi_p^{(1)}, \varphi_p^{(2)})$ to be a matching pair in Propositions 3.1 and 3.3.*

Then $(\varphi^{(1)}, \varphi^{(2)})$ is a matching pair, and

$$I(g, \varphi^{(1)}) = I(g, \varphi^{(2)}), \quad g \in \text{SL}_2(\mathbb{A}).$$

In next two sections, we will give arithmetic and interpretations of the theta integrals in some special cases.

4. DEFINITE QUATERNIONS, REPRESENTATIONS NUMBERS, AND SUPERSINGULAR ELLIPTIC CURVES

We first review a general fact about positive definite quadratic forms for the convenience of the reader. Let (V, Q) be a positive definite quadratic space of even dimension m . Define

$$\varphi_\infty(x) = e^{-2\pi Q(x)} \in S(V_\infty).$$

Then it has the properties

$$\varphi_\infty(hx) = \varphi_\infty(x), \quad \omega(k_\theta)\varphi_\infty = e^{\frac{m}{2}i\theta}\varphi_\infty$$

for $h \in O(V)(\mathbb{R})$ and $k_\theta \in \text{SO}_2(\mathbb{R}) \subset \text{SL}_2(\mathbb{R})$. For any $\varphi_f \in S(\hat{V})$, where $\hat{V} = V \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, the theta kernel

$$\theta(\tau, h, \varphi_f \varphi_\infty) = v^{-\frac{m}{2}} \theta(g_\tau, h, \varphi_f \varphi_\infty)$$

is a holomorphic modular form of weight $\frac{m}{2}$ for some congruence subgroup, so is

$$I(\tau, \varphi_f \varphi_\infty) = v^{-\frac{m}{2}} I(g_\tau, \varphi_f \varphi_\infty).$$

Here $g_\tau = n(u)m(\sqrt{v})$ for $\tau = u + iv \in \mathbb{H}$.

For an even integral lattice L of V , we denote

$$(4.1) \quad \theta(\tau, L) = \theta(\tau, \text{char}(\hat{L})\varphi_\infty), \quad I(\tau, L) = I(\tau, \text{char}(\hat{L})\varphi_\infty).$$

Recall that two lattices L_1 and L_2 of V are equivalent if there is $h \in O(V)$ such that $hL_1 = L_2$. Two lattices L_1 and L_2 are in the same genus if they are equivalent locally everywhere, i.e., there is $h \in O(V)(\hat{\mathbb{Q}})$ such that $h\hat{L}_1 = \hat{L}_2$. We also recall that $O(V)(\mathbb{A})$ acts on the set of lattices as follows: $hL = (h_f \hat{L}) \cap V$ where h_f is the finite part of $h = h_f h_\infty$. Let $\text{gen}(L)$ be the genus of L —the set of equivalence classes of lattices in the same genus of L . Then the above discussion implies that

$$O(V)(\mathbb{Q}) \backslash O(V)(\mathbb{A}) / K(L) O(V)(\mathbb{R}) \cong \text{gen}(L), \quad [h] \mapsto hL,$$

where $K(L)$ is the stabilizer of \hat{L} in $O(V)(\hat{\mathbb{Q}})$.

Proposition 4.1. *Let*

$$r_L(n) = |\{x \in L : Q(x) = n\}|, \quad r_{\text{gen}(L)}(n) = \left(\sum_{L' \in \text{gen}(L)} \frac{1}{|O(L')|} \right)^{-1} \sum_{L' \in \text{gen}(L)} \frac{r_{L'}(n)}{|O(L')|},$$

where $O(L)$ is the stabilizer of L in $O(V)$. Then ($q = e(\tau)$)

$$\begin{aligned} \theta(\tau, h, L) &= \sum_{n=0}^{\infty} r_{hL}(n) q^n, \\ I(\tau, L) &= \sum_{m=0}^{\infty} r_{\text{gen}(L)}(m) q^m. \end{aligned}$$

Proof. (sketch) This is well-known and we sketch the main steps for the convenience of the reader. The formula for θ follows directly from the definition. For theta integral, notice that $\text{char}(\hat{L})$ is $K(L)$ -invariant. Write

$$O(V)(\mathbb{A}) = \cup_{j=1}^r O(V)(\mathbb{Q}) h_j K(L) O(V)(\mathbb{R}).$$

Then $\text{gen}(L) = \{h_j L : j = 1, \dots, r\}$, and

$$\begin{aligned} \text{vol}([O(V)]) I(\tau, L) &= \sum_j \theta(\tau, h_j, L) \int_{(h_j^{-1} O(V)(\mathbb{Q}) h_j) \cap K(L) \backslash K(L) O(V)(\mathbb{R})} 1 dh \\ &= \text{vol}(K(L) O(V)(\mathbb{R})) \sum_j \frac{\theta(\tau, h_j, L)}{|O(h_j L)|} \\ &= \text{vol}(K(L) O(V)(\mathbb{R})) \sum_{n=0}^{\infty} \left(\sum_{L' \in \text{gen}(L)} \frac{r_{L'}(n)}{|O(L')|} \right) q^n. \end{aligned}$$

On the other hand, the same calculation gives

$$\text{vol}([O(V)]) = \text{vol}(K(L) O(V)(\mathbb{R})) \sum_{L' \in \text{gen}(L)} \frac{1}{|O(L')|}.$$

This proves the formula for $I(\tau, L)$. □

Proof of Theorem 1.1: Let $V(D)$ be the quadratic space associated to the quaternion algebra $B(D)$ of discriminant D . Recall that a Eichler order of conductor N , denoted by $\mathcal{O}_D(N)$, is an order O of $B(D)$ such that

- (1) When $p|D$, $O_p := O \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is the maximal order of $B(D)_p = B_p^{ra}$.
- (2) When $p \nmid D\infty$, there is an identification $B(D)_p \cong M_2(\mathbb{Q}_p)$ under which $\mathcal{O}_D(N)_p = R_p(N)$. Here

$$R_p(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) : c \equiv 0 \pmod{N} \right\}.$$

Now let $V^{(1)} = V(Dp)$ and $V^{(2)} = V(Dq)$ with D satisfying the condition in the theorem. Then $V^{(i)}$ are both positive definite. Define $\varphi^{(1)} = \prod_l \varphi_l^{(1)} \in S(V^{(1)}(\mathbb{A}))$ as follows.

$$\varphi_l^{(1)} = \begin{cases} \varphi_\infty^{ra} & \text{if } l = \infty, \\ \text{char}(R_l(N)) & \text{if } l \nmid Dq, \\ \varphi_l^{ra} & \text{if } l \mid Dp, \\ \frac{-2}{q-1} \varphi_{q,0}^{sp} + \frac{q+1}{q-1} \varphi_{q,1}^{sp} & \text{if } l = q \end{cases}$$

where $\varphi_{l,i}^{sp}$ and φ_l^{ra} are the functions defined in (3.1) with added subscript l (to indicate its independence). Then one has

$$\varphi_f^{(1)} = \frac{-2}{q-1} \text{char}(\hat{\mathcal{O}}_{Dp}(N)) + \frac{q+1}{q-1} \text{char}(\hat{\mathcal{O}}_{Dp}(Nq)).$$

So

$$I(\tau, \varphi^{(1)}) = \frac{-2}{q-1} I(\tau, \mathcal{O}_{Dp}(N)) + \frac{q+1}{q-1} I(\tau, \mathcal{O}_{Dp}(Nq)).$$

One defines $\varphi^{(2)}$ the same way with the roles of p and q switched. Then $\varphi^{(1)}$ and $\varphi^{(2)}$ form a matching pair by Proposition 3.3. So Proposition 3.5 implies

$$I(\tau, \varphi^{(1)}) = I(\tau, \varphi^{(2)})$$

and thus

$$\frac{-2}{q-1} I(\tau, \mathcal{O}_{Dp}(N)) + \frac{q+1}{q-1} I(\tau, \mathcal{O}_{Dp}(Nq)) = \frac{-2}{p-1} I(\tau, \mathcal{O}_{Dq}(N)) + \frac{p+1}{p-1} I(\tau, \mathcal{O}_{Dq}(Np)).$$

Taking m -th Fourier coefficients, one proves the theorem.

The case $D = 1$ has special geometric meaning as indicated in the introduction. Let $Y_0(N)$ be the moduli stack of pairs (E, C) where E is an elliptic curve and C is a cyclic sub-scheme of order N . It is regular and flat over \mathbb{Z} and smooth over $\mathbb{Z}[\frac{1}{N}]$. For a prime $p \nmid N$, let $SS_p(N)$ be the supersingular locus of $Y_0(N)(\bar{\mathbb{F}}_p)$ —the $\bar{\mathbb{F}}_p$ -point (E, C) such that E is supersingular, i.e. $\text{End}(E)$ and $\text{End}(E')$ are maximal orders of $B(p)$. In this case, the endomorphism ring $\text{End}(E, C)$ is an Eichler order $\mathcal{O}_p(N)$ of conductor N . Every Eichler of $B(p)$ comes this way. For two points $(E_1, C_1), (E_2, C_2) \in SS_p(N)$, $\text{Hom}((E_1, C_1), (E_2, C_2))$, which consists of isogenies $(f : E_1 \rightarrow E_2)$ with $f(C_1) \subset C_2$, is a quadratic lattice with respect to $\deg f$, and is in the same genus of $\text{End}(x_1)$ and $\text{End}(x_2)$. One can actually prove (see example [Ya2]) that all $\text{Hom}(x_1, x_2)$ form the genus of $L = \mathcal{O}_p(N)$ as x_1 and x_2 runs through the supersingular locus $SS_p(N)$. So we have

Proposition 4.2. *One has*

$$r_{p,N}(m) = \left(\sum_{x_1, x_2 \in SS_p(N)} \frac{1}{|\text{Aut}(x_1)| |\text{Aut}(x_2)|} \right)^{-1} \sum_{x_1, x_2 \in SS_p(N)} \frac{r_{\text{Hom}(x_1, x_2)}(m)}{|\text{Aut}(x_1)| |\text{Aut}(x_2)|}.$$

5. INDEFINITE QUATERNIONS AND SHIMURA CURVES

Associated to a square-free integer $D > 0$ with even number of prime factors, is an indefinite quaternion algebra $B = B(D)$ of discriminant D . In particular, We choose and fix an embedding $i : B \hookrightarrow B_\infty \cong M_2(\mathbb{R})$ such that the inner isomorphism $X \mapsto wXw^{-1}$ preserves $i(B)$. We denote \det for the reduced norm on B , then $V = V(D) = (B, \det)$ is of signature $(2, 2)$ and is anisotropic when $D > 1$. According to [Ku2, Theorem 4.23], the theta integral $I(g, \varphi)$ in Proposition 3.5 is a generating function of degrees of some divisors with respect to the tautological line bundle over the Shimura variety associated to V . In our case, the line bundle can be identified with the line bundle of two variable modular forms of weight 1, and the divisors can be identified with Hecke correspondences on a Shimura curve as we will see now.

The action of $B^\times \times B^\times$ on V via

$$(g_1, g_2)X = g_1 X g_2^{-1}$$

gives an identification of $\mathrm{GSpin}(V)$ with

$$H = \{(g_1, g_2) \in B^\times \times B^\times : \det g_1 = \det g_2\}.$$

The associated spin norm is $\mu(g_1, g_2) = \det g_1$. It has the exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow H \rightarrow \mathrm{SO}(V) \rightarrow 1.$$

Let \mathbb{D} be the Hermitian domain of oriented negative 2-planes in $V_{\mathbb{R}}$, and

$$\mathcal{L} = \{w \in V_{\mathbb{C}} = M_2(\mathbb{C}) : (w, w) = 0, (w, \bar{w}) < 0\}$$

on both of which $H(\mathbb{R})$ acts naturally. The map

$$f : \mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}, \quad w = u + iv \mapsto \mathbb{R}(-u) + \mathbb{R}v$$

gives an $H(\mathbb{R})$ -equivariant isomorphism between $\mathcal{L}/\mathbb{C}^\times$ with \mathbb{D} . Thus \mathcal{L} is a (tautological) line bundle over \mathbb{D} . The Hermitian domain has also a tube representation which we will need. Indeed, the map

$$\mathbf{w} : (\mathbb{H})^2 \times (\mathbb{H}^-)^2 \rightarrow \mathcal{L}, \quad \mathbf{w}(z_1, z_2) = \begin{pmatrix} z_1 z_2 & z_1 \\ z_2 & 1 \end{pmatrix},$$

gives an isomorphism

$$(\mathbb{H})^2 \times (\mathbb{H}^-)^2 \cong \mathcal{L}/\mathbb{C}^\times \cong \mathbb{D}.$$

We will identify \mathbb{D} with $(\mathbb{H})^2 \times (\mathbb{H}^-)^2$ via this isomorphism. The natural action of $B^\times \times B^\times$ on V induces the following action on $(\mathbb{H})^2 \times (\mathbb{H}^-)^2$:

$$(5.1) \quad (g_1, g_2)(z_1, z_2) = (i(g_1)z_1, i(g_2)^*z_2)$$

where $g^* = {}^t g^{-1}$ for $g \in \mathrm{GL}_2(\mathbb{R})$. One also has

$$(5.2) \quad (g_1, g_2)w(z_1, z_2) = w((g_1, g_2)(z_1, z_2))(c_1 z_1 + d_1)(c_2 z_2 + d_2)$$

for

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in H(\mathbb{R}), \quad g_2^* = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in H(\mathbb{R})$$

Associated to a compact open subgroup K of $H(\hat{Q})$ is a Shimura variety X_K over \mathbb{Q} such that

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash \mathbb{D} \times H(\hat{Q}) / K.$$

Moreover, \mathcal{L} descends to a line bundle on X_K , which we continue to denote by \mathcal{L} . It can be identified with the line bundle of two variable modular forms of weight $(1, 1)$. In this section, we always assume

$$K = \{(k_1, k_2) \in \hat{\mathcal{O}}_D(N)^\times \times \hat{\mathcal{O}}_D(N)^\times : \det k_1 = \det k_2\} \subset H(\hat{Q})$$

which preserves the lattice $L = \mathcal{O}_D(N)$.

Lemma 5.1. *Let the notation be as above. Then one has an isomorphism*

$$X_0^D(N) \times X_0^D(N) \cong X_K, ([z_1], [z_2]) \mapsto [z_1, wz_2]$$

where $X_0^D(N) = \Gamma_0^D(N) \backslash \mathbb{H}$ is the Shimura curves defined in the introduction (recall $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$).

Proof. Let

$$H_1 = \{(g_1, g_2) \in H : \det g_1 = \det g_2 = 1\} = \ker \mu = \text{Spin}(V), \quad K_1 = H_1(\hat{Q}) \cap K.$$

By the strong approximation theorem, one has

$$H_1(\mathbb{A}) = H_1(\mathbb{Q}) K_1 H_1(\mathbb{R}).$$

Since $\mu(H(\mathbb{Q})KH(\mathbb{R})^+) = \mathbb{A}^\times$, one has then

$$H(\mathbb{A}) = H(\mathbb{Q})KH(\mathbb{R})^+.$$

So

$$\begin{aligned} X_K &= H(\mathbb{Q}) \backslash H(\mathbb{A}) / (KK_\infty) \\ &= H(\mathbb{Q}) \backslash (H(\mathbb{Q})KH(\mathbb{R})^+) / (KK_\infty) \\ &= (H(\mathbb{Q}) \cap (KH(\mathbb{R})^+)) \backslash H(\mathbb{R})^+ / K_\infty. \end{aligned}$$

Here K_∞ is stabilizer of $(i, i) \in H^2$ in $H(\mathbb{R})$ and also in $H(\mathbb{R})^+$. Notice that

$$H(\mathbb{Q}) \cap (KH(\mathbb{R})^+) = H_1(\mathbb{Q}) \cap K_1 = \Gamma_0^D(N) \times \Gamma_0^D(N).$$

So

$$X_K = X_0^D(N) \times X_0^D(N)^*,$$

where $X_0^D(N)^* = \Gamma_0^D(N) \backslash \mathbb{H}$ with a slightly different action $\gamma * z = \gamma^*(z)$. Now the lemma follows from the isomorphism

$$X_0^D(N) \cong X_0^D(N)^*, \quad [z] \mapsto [wz].$$

□

Let

$$\Omega = -\frac{1}{4\pi} (y_1^{-2} dx_1 \wedge dy_1 + y_2^{-2} dx_2 \wedge dy_2)$$

be as in [Ku2, Example 4.13]. It corresponds to Chern class $-c_1(\mathcal{L})$ in $H^2(X_K)$.

Next, we describe the Kudla cycle on X_K and relate it to Hecke correspondence on $X_0^D(N)$. Let $\Omega_0 = -\frac{1}{4\pi} y^{-2} dx \wedge dy$ be as in the introduction, and let π_1 and π_2 be two natural projections of $X_K = X_0^D(N) \times X_0^D(N)$ onto $X_0^D(N)$. Then

$$\Omega = (\pi_1^*(\Omega_0) + \pi_2^*(\Omega_0)).$$

Moreover, one has by [KRY, (2.7)] and [Mi, Lemma 5.3.2]

$$\begin{aligned} (5.3) \quad \text{vol}(X_0^D(N), \Omega_0) &:= \int_{X_0^D(N)} \Omega_0 = [\mathcal{O}_D^1 : \Gamma_0^D(N)] \zeta_D(-1) \\ &= -\frac{DN}{12} \prod_{p|N} (1+p^{-1}) \prod_{p|D} (1-p^{-1}) \in \frac{1}{12} \mathbb{Z}_{<0}, \end{aligned}$$

where $\zeta_D(s) = \prod_{p \nmid D} (1-p^{-s})^{-1}$ is the partial zeta function, and \mathcal{O}_D is a maximal order of B containing $\mathcal{O}_D(N)$.

For an $x \in V$ with $\det(x) > 0$ and $h \in H(\hat{Q})$, x^\perp is of signature $(1, 2)$ and defines a sub-Shimura variety $Z(x)$ of $X_{hKh^{-1}}$, its right translate by h gives a divisor $Z(x, h)$ in X_K . For $\varphi_f \in S(\hat{V})^K$ and $m \in \mathbb{Q}_{>0}$, one defines the associated Kudla cycle $Z(m, \varphi_f)$ as

$$Z(m, \varphi_f) = \sum_{j=1}^r \varphi_f(h_j^{-1}x) Z(x, h_j)$$

if there is $x \in V$ such that $\det(x) = m$ and

$$\text{Supp}(\varphi_f) \cap \{x \in V(\hat{Q}) : \det x = m\} = \coprod_{j=1}^r Kh_j^{-1}x.$$

Otherwise, we defines $Z(m, \varphi_f) = 0$.

Lemma 5.2. *Let $T_{D,N}(m)$ be the Hecke operator on $X_0^D(N)$ as in the introduction. Then (under the identification $X_K \cong X_0^D(N) \times X_0^D(N)$ in Lemma 5.1)*

$$Z(m, \text{char}(\hat{L})) = T_{D,N}(m)$$

where $L = \mathcal{O}_D(N)$.

Proof. Let $L_m = \{x \in L : \det x = m\}$. By proof of Lemma 5.1, one has $H(\hat{Q}) = H(\mathbb{Q})K$. So in the decomposition ($x \in V$ with $Q(x) = m$)

$$\hat{L}_m = \coprod Kh_j^{-1}x$$

we may assume $h_j \in H(\mathbb{Q})$. This implies

$$L_m = \coprod \Gamma_K h_j^{-1}x = \coprod \Gamma_K x_j, \quad x_j = h_j^{-1}x \in L,$$

where $\Gamma_K = K \cap H(\mathbb{Q})$, and

$$Z(m, \varphi_f) = \sum_j Z(x, h_j) = \sum_j Z(h_j^{-1}x) = \sum_j Z(x_j) = \Gamma_K \backslash \mathbb{D}_m.$$

where \mathbb{D}_m is the set of $(z_1, z_2) \in \mathbb{H}^2 \times (\mathbb{H}^-)^2$ which satisfying $z_1 = x(wz_2)$ for some $x \in L_m$.

Since there is some $(\gamma_1, \gamma_2) \in \Gamma_K$ with $\det \gamma_1 = \det \gamma_2 = -1$, one has thus

$$Z(m, \varphi_f) = (\Gamma_0^D(N) \times \Gamma_0^D(N)) \backslash \mathbb{D}_m^+ = T_{D,N}(m).$$

Here $\mathbb{D}_m^+ = (\mathbb{H} \times \mathbb{H}) \cap \mathbb{D}_m$. This proves the lemma. \square

Theorem 5.3. *For $\varphi_f = \text{char}(\hat{\mathcal{O}}_D(N))$, one has*

$$I(\tau, \varphi_f \varphi_\infty^{sp}) = v^{-1} I(g_\tau, \varphi_f \varphi_\infty^{sp}) = \sum_{m=0}^{\infty} r'_{D,N}(m) q^m$$

where $r'_{D,N}(0) = 1$, and for $m > 0$

$$r'_{D,N}(m) = \frac{1}{\text{vol}(X_0^D(N), \Omega_0)} \deg T_{D,N}(m)$$

as in the introduction.

Proof. Write

$$I(\tau, \varphi_f \varphi_\infty^{sp}) = \sum_{m=0}^{\infty} c(m) q^m.$$

By [Ku2, Section 4.8], one has $c(0) = 1$ and for $m > 0$

$$c(m) = (\text{vol}(X_K, \Omega^2))^{-1} \int_{Z(m, \varphi_f)} \Omega.$$

Clearly,

$$\text{vol}(X_K, \Omega^2) = \frac{1}{2} \frac{1}{4\pi^2} \int_{X_0^D(N) \times X_0^D(N)} \frac{dx_1 \wedge dy_1}{y_1^2} \wedge \frac{dx_2 \wedge dy_2}{y_2^2} = \frac{1}{2} \text{vol}(X_0^D(N), \Omega_0)^2.$$

On the other hand, $\Omega = \pi_1^*(\Omega_0) + \pi_2^*(\Omega_0)$. So Lemmas 5.2 gives

$$\begin{aligned} c(m) &= \int_{T_{D,N}(m)} (\pi_1^*(\Omega_0) + \pi_2^*(\Omega_0)) \\ &= 2 \int_{T_{D,N}(m)} \pi_1^*(\Omega_0) \\ &= 2 \deg T_{D,N}(m) \int_{X_0^D(N)} \Omega_0. \end{aligned}$$

So $c(m) = r'_{D,N}(m)$ as claimed. \square

Proof of Theorems 1.2, 1.3 and 1.4: Now Theorems 1.2, 1.3 and 1.4 follows the same way as Theorem 1.1. We verify Theorem 1.3 and leave the others to the reader. Let $V^{(1)} = V(D)$ and $V^{(2)} = V(Dp)$ as in the notation of proof of Theorem 1.1, and let $\varphi^{(i)} = \prod_l \varphi_l^{(i)} \in S(V^{(i)}(\mathbb{A}))$ be as follows. For $l \nmid p\infty$, we identify $\mathcal{O}_D(N)_l$ with $\mathcal{O}_{Dp}(N)_l$ and define $\varphi_l^{(i)} = \text{char}(\mathcal{O}_D(N)_l)$. Let

$$\varphi_\infty^{(1)} = \varphi_\infty^{ra}, \quad \varphi_\infty^{(2)} = \varphi_\infty^{sp}.$$

Finally, let

$$\varphi_p^{(1)} = -\frac{2}{p-1}\varphi_{p,0}^{sp} + \frac{p+1}{p-1}\varphi_{p,1}^{sp}, \quad \varphi_p^{(2)} = \varphi_p^{ra}.$$

Then $\varphi^{(1)}$ and $\varphi^{(2)}$ match by the results in Section 3. So one has by Proposition 3.5

$$I(\tau, \varphi^{(1)}) = I(\tau, \varphi^{(2)}).$$

Comparing the m -coefficients of the both sides, one proves Theorem 1.3.

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